Externalities in the games over electrical power transmission networks*

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Abstract

An electrical transmission network consists of producers, consumers and the power lines connecting them. We build an ideal (lossless) DC load flow model as a cooperative game over a graph with the producers and consumers located at the nodes, each described by a maximum supply or desired demand and the power lines represented by the edges, each with a given power transmission capacity and admittance value describing its ability to transmit electricity.

Today's transmission networks are highly interconnected, but organisationally partitioned into several subnetworks, the so-called *balancing groups* with balanced production and consumption. We study the game of balancing group formation and show that the game contains widespread externalities that can be both negative and positive. We study the stability of the transportation network using the recursive core. While the game is clearly cohesive, we demonstrate that it is not necessarily superadditive. We argue that subadditivity may be a barrier to achieve full cooperation. Finally the model is extended to allow for the extension of the underlying transmission network.

Keywords and phrases: Energy transmission networks, Cooperative game theory, Partition function form games, Externalities

JEL-codes: C71, L14, L94

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1 Introduction

All complex systems (Érdi, 2008) can be examined from two distinguished points of view: the flow of energy, and the flow of information. From the smallest size, such as a cell, where these considerations define biochemical metabolic and signaling networks to continent-wide social or economical networks, where both the energy and information flow takes place in many forms, the subsystems controlling these distinguished flows have different tasks, but they are interdependent. One of the key subsystems of the global or continental energy flow is the electrical power transmission network. We study the electrical energy market as an interaction of market participants, where the possible interactions are constrained by laws of physics and market regulations.

Models of the *power grid* derived from engineering principles are appropriate to describe the physical nature of energy transfer, and they can be used to find appropriate technological solutions during the construction, modification and operation of these systems. These models, however, tend to neglect the motivations of the participants as economic agents. Economics models, in contrast, focus on the financial considerations, ignoring the problems and limitations originating from the physical details of the system. We suppose, that there are phenomena, so called *emergent properties* (Érdi, 2008), which can be examined only beyond the certain level of complexity. Such an example is the influence of multiple power plants of the market belonging to the same company, or circle of interest. We study the interaction between the physical and economic aspects of the system focussing on the incentives for group formation.

The aim of this paper is therefore to define a framework, which is capable of describing the interactions in the electrical energy market in a simplified way, considering both physical and economical aspects of the participants and the network itself. Furthermore, we try to fulfill these aims with the methodology of cooperative game theory, which approach has already been successfully used in the case of many economic systems.

The structure of the paper will, accordingly, be as follows: First we describe the physical properties of the network and derive a simplified framework that describes the stakeholders' characteristics and utilities as well as the partition function form approach, a model in cooperative game theory that can handle coalition formation with widespread externalities. The main part of the paper is Section 3, where, by means of a series of examples we demonstrate some unexpected properties that might provide incentives for network participants to go against the usual trends of network development such as increased levels of integration. We close with a brief summary a set of open questions.

2 Materials and Methods

2.1 Background literature

When studying electric power transmission networks, most of the research in economics has been on the topics of competition, market power and regulation (see, for example Cardell, Hitt, and Hogan, 1997; Gilbert, Neuhoff, and Newbery, 2004; Neuhoff, Barquin, Boots, Ehrenmann, Hobbs, Rijkers, and Vázquez, 2005; Chen, Hobbs, Leyffer, and Munson, 2006) and very few works study the market and the transmission issues in their whole complexity (Kirschen and Strbac, 2004, is a notable exception). Hobbs (1992); Bai, Shahidehpour, and Ramesh (1997) use already game theory for transmission analysis, and Orths, Schmidtt, Styczynski, and Verstege (2001) describes a game theoretic approach of a multi-criteria optimization problem related to transmission planning and operation. A strategic gaming approach is described in Kleindorfer, Wu, and Fernando (2001).

Gately (1974) is probably the first to apply cooperative game theory to planning investments of electrical power systems. Evans, Zolezzi, and Rudnick (2003) describes a cost assignment model for electrical transmission system expansion using Kernel theory. Contreras (1997) provides a decentralized framework in his thesis to study the transmission network expansion problem using cooperative game theory.

2.2 Model of the energy transmission network

Before presenting our model first we summarise the main assumptions and constraints regarding the description of the power transmission network. The notations and the mathematical formalism are based on Oren, Spiller, Varaiya, and Wu (1995) and Contreras (1997).

We assume that the power transmission system is described by a graph, the *system graph*, in which n nodes (or buses) are connected by m edges, which naturally represent the transmission lines.

As foreshadowed, generators can be characterized by the quantity of actual and maximal generated (or supplied) power, while consumers are described by the amount of actually and ideally consumed power. We assume that a transmission line is characterized by its admittance value, denoted by Y_{ij} (which will be equal to susceptance in this case, for we neglect the real part of impedance values), and maximum transmission capacity (or branch power flow limit) $\bar{q}_{ij}$.

According to our modelling considerations, we describe the voltage at node i with sinusoidal waveform:

$$v_i(t) = V_i \sin(\omega t + \theta_i) \tag{1}$$

where V_i stands for the magnitude, $\omega = 2\pi f$ denotes the frequency in rad/s and θ_i is the phase angle.

If we assume that the nodes i and j are connected by a transmission line with admittance $Y_{ij} = Y_{ji}$, the (real) power flow from i to j can be described with:

$$q_{ij} = V_i V_j Y_{ij} \sin(\theta_i - \theta_j) \tag{2}$$

We use the sign convention $q_{ij} > 0$ if the power flows from i to j. $q_{ij} = -q_{ji}$ We can formalize the energy conservation for each node as follows. The net power q_i injected into (or drawn from) the network at bus i addition to the total inflow is equal to the total outflow:

$$q_i = \sum_{j=1}^{n} q_{ij} \tag{3}$$

Without the loss of generality, let us assume $V_i \equiv 1$. in this case

$$q_i = \sum_{j=1}^{n} Y_{ij} \sin(\theta_i - \theta_j) \tag{4}$$

which means $n-1$ independent equations (as $q_1 + ... + q_n = 0$). Let us choose $\theta_n \doteq 0$. In this case the individual line flows can be expressed as:

$$q_{ij} = Y_{ij} \sin(\theta_i - \theta_j) \tag{5}$$

Assuming that $(\theta_i - \theta_j)$ is small, $\sin(x)$ may be approximated with x. This leads to the so called "DC load flow model", which exhibits the following uniqueness property: *Given power injections and power consumptions at each node, the phase angles θ_i are determined by solving a system of linear equations. From the phase angle differences, the line flows can be uniquely determined.*

We can summarize the equations in the following matrix formalism (Contreras, 1997): The relation between the total inlet/outlet power and power flows can be described by

$$AQ = P \tag{6}$$

where $A \in \mathbb{R}^{n\times m}$ is the Node-branch incidence matrix of the network, $Q \in \mathbb{R}^m$ denotes the power flow vector, and $P \in \mathbb{R}^n$ is the power injection vector (composed of $[q_1, q_2...]$). If we substitute the individual power flows in Equation 6 with the linearized expressions from Equation 2, we can write

$$B(Y)\Theta = P \tag{7}$$

where $B(Y) \in \mathbb{R}^{n\times n}$ denotes the susceptance matrix whose elements are $B_{kl} = -Y_{kl}$ for the off-diagonal terms and

$$B_{kk} = \sum_{l\neq k \in \Psi} B_{kl}$$

(the column sum of off-diagonals) for diagonal elements (where Ψ is the actual column). $\Theta \in \mathbb{R}^n$ is vector of nodal voltage angles.

The constraint describing the maximum line power flows can be derived as

$$|Q| = |B^D A^T \Theta| < \bar{Q} \tag{8}$$

where $|\bar{Q}|$ is branch power flow limit vector (composed of the elements $\bar{q}_{ij}$), and B^D is a diagonal matrix with $B^D_{kk} = Y_{ij}$.

2.2.1 The linear programming form of the generator rescheduling problem

As we know from Equation 7, $B\Theta = P$. The matrix B is singular due to the column conservation property, but since in the calculation of flows only the differences of the elements of the vector Θ are appearing (see Equation 2), we may express it as

$$\Theta = B^+ P \tag{9}$$

where B^+ is the Moore-Penrose pseudoinverse of B. Constraint 8 becomes

$$|B^D A^T \Theta| = |B^D A^T B^+ P| < \bar{Q} \tag{10}$$

Let us suppose the initial power generation/consumption vector (of which's elements are positive in the case of consumers and negative in the case of generators) is equal to P^{init} and let us determine the initial flows Q^{init}. We define a sign vector s_P and a diagonal matrix s_Q^D, corresponding to the signs of flows.

$$s_P = -\text{sign}(P^{\text{init}}) \qquad s_Q^D = \text{diag}(\text{sign}(Q^{\text{init}})) \tag{11}$$

As we know, the general form of linear programming (LP) problem is

$$\min_x f^T x \qquad \text{subject to: } A_{\text{ineq}} x \leqq b_{\text{ineq}}, \quad A_{\text{eq}} x = b_{\text{eq}}. \tag{12}$$

We may put the rescheduling problem in an LP form, as follows

$$f = s_P \quad A_{\text{ineq}} = s_Q^D B^D A^T B^+ \quad b_{\text{ineq}} = \bar{Q} \quad A_{\text{eq}} = [1 \;\; 1 ... 1] \quad b_{\text{eq}} = 0 \tag{13}$$

where the inequality constraints are corresponding to the maximum load of lines and the equality constraints corresponding to the balance of total inlet and outlet power. Further linear constraints can be added to the problem, describing the minimal and maximal values of nodal power values, corresponding to maximum generator capacity, and minimum consumption at certain nodes.

2.3 The cooperative game on the transmission network

We define a game in partition function form (PFF, Thrall and Lucas, 1963) on the transmission system model. A PFF game is a pair (N, V), where N is the set of players, and $V : \Pi \to (2^N \to \mathbb{R})$ is the partition function, which assigns characteristic functions $(v : 2^N \to \mathbb{R})$ to each partition $\mathcal{P} \in \Pi(N)$ (where $\Pi(N)$ denotes the set of partitions of N).

We will make the following assumptions regarding the game to be defined:

- An initial configuration of the network is given with generation and consumption values.
- The overall power inlet/outlet of any coalition has to be in balance (this can be implemented in the LP formalism, as defining additional equality constraints describing the coalitional balances - which practically means additional rows in the matrix A_{eq}). This implies that every non trivial coalition must hold at least one generator and one consumer.
- We assume an independent network regulator, who determines the possible inflow/outflow quantities, according to the maximum possible overall consumption (for a given coalition structure, and the implied constraints, the amount of the *total* transmitted energy is optimized).
- Every generator produces (sells) as much energy as possible, and the consumers are interested in consuming their ideal amount. We will define the function μ as $\mu(i) = q_i$ for both generators and consumers, assuming that consumers can not consume more than their ideal amount, nor can generators produce more than their maximal capacity. The value of the characteristic function for a given coalition C is the sum of its members' utilities:

$$v(C) = \sum_{i \in C} \mu(i)$$

During any change of the coalition structure, which implies the change of network constraints, the generators will be rescheduled, in order to reach the maximal transmission capacity of the network under the limitations implied by the actual coalition structure.

2.3.1 The core

The production and consumption level of the network is determined by the network manager, who simply runs an optimisation process given the partition of the players into balancing groups and the characteristics of the network. The partition into balancing groups may be exogenous, but in a liberalised market it is natural to assume that nodes are free to leave their balancing group, join another or even that a group of players forms a new balancing group altogether. Note that the restructuring of the nodes leaves the power grid, that is, the underlying network unaffected.

Partitions where there are no incentives for such restructuring, where no coalition of players will benefit from the formation of another coalition deserve special attention. The core Gillies (1959) collects imputations of a characteristic function form game where no coalition can benefit from deviating. In a PFF game whether a coalition benefits from deviating depends on the induced partition of the players. The $\alpha-$core Aumann and Peleg (1960) assumes that a coalition deviates only if it gets a higher payoff irrespective of the induced partition. In the $\gamma-$core Chander and Tulkens (1997) the coalition must face individually best responses. Here we use the concept of the *recursive core* Kóczy (2007, 2009), that allows the remaining, residual players to freely react and form a core-stable partition before the payoff of the deviating coalition is evaluated. In the following we recall the definition.

First we define the *residual game* over the set $R \subsetneq N$. Assume $N \setminus R$ have formed $\mathcal{P}_{N\setminus R}$. Then the residual game $(R, V_{\mathcal{P}_{N\setminus R}})$ is the PFF game over the player set R with the partition function given by $V_{\mathcal{P}_{N\setminus R}}(C, \mathcal{P}_R) = V(C, \mathcal{P}_R \cup \mathcal{P}_{N\setminus R})$.

Definition 1 ((Pessimistic) recursive core) *Let (N, V) be a PFF game.*

1. *Trivial game. The* core *of $(\{1\}, V)$ is the only outcome with the trivial partition:*
$$C(\{1\}, V) = \left\{ \left(V\left(1, (1)\right), (1) \right) \right\}.$$
2. *Inductive assumption. Assume that the core $C(R, V)$ has been defined for all games with at most $k-1$ players. The* assumption about game

(R, V) *is*

$$A(R, V) = \begin{cases} C(R, V) & \text{if } C(R, V) \neq \varnothing \\ \Omega(R, V) & \text{otherwise.} \end{cases}$$

where $\Omega(R, V)$ *denotes the set of outcomes in* (R, V).

3. *Dominance. The outcome* $(x, \mathcal{P})$ *is* dominated via the coalition S forming partition $\mathcal{P}_S$ *if for **all*** $(y_{N \setminus S}, \mathcal{P}_{N \setminus S}) \in A(N \setminus S, V_{\mathcal{P}_S})$ *there exists an outcome* $((y_S, y_{N \setminus S}), \mathcal{P}_S \cup \mathcal{P}_{N \setminus S}) \in \Omega(N, V)$ *such that* $y_S > x_S$. *The outcome* $(x, \mathcal{P})$ *is* dominated *if it is dominated via a coalition.*

4. *Core. The* core, *denoted* $C(N, V)$, *is the set of undominated outcomes.*

The (pessimistic) core is denoted $C(N, V)$.

The recursive core is well-defined, though it may be empty.

3 Results

3.1 Negative and positive externalities

First, in this section we will demonstrate the emergence of negative and positive externalities on a simple 5 node network depicted in Fig. 1.

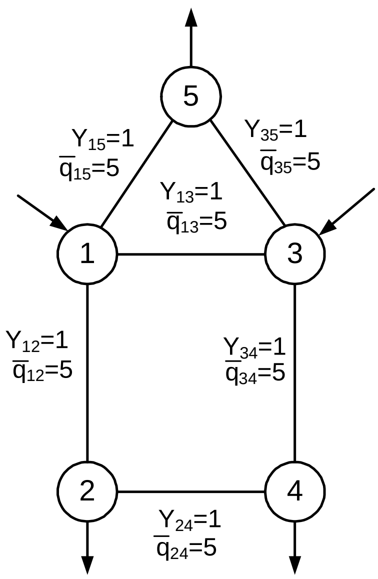

Figure 1: The basic structure and line parameters of the 5 node network.

Let us assume that the maximal capacity of the generators 1 and 3 are 10 and 20 units, and that the ideal consumption amount of the consumers $2, 4$ and 5 are $7, 5, 10$ respectively.

3.1.1 Flows in the case of the coalition structure $\{1, 2\}, 3, 4, 5$

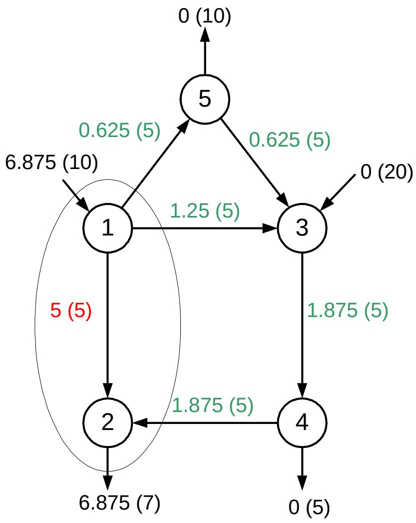

Figure 2: Flows in the case of the coalition structure $\{1, 2\}, 3, 4, 5$. The numbers on the edges denote the actual flows, while the numbers in parentheses denote the flow limits. Red labels on the edges denote flow rates equal to maximum transmission capacity. The numbers on the edges denote the actual and maximum (parentheses) injected power in the case of generators, and the actual and ideal (parentheses) consumed amount in the case of consumers.

The flows corresponding to the maximal total transmitted energy amount is depicted in Fig. 2.

We consider this scenario as a reference case where the players $3, 4$ and 5 form trivial one-member coalitions. We can easily derive the value of the function μ, and so the characteristic function v:

$$\mu(1) = 6.875 \quad \mu(2) = 6.875 \quad \mu(3) = 0 \quad \mu(4) = 0 \quad \mu(5) = 0 \tag{14}$$

$$v(\{1, 2\}) = 6.875 + 6.875 = 13.75 \quad v(3) = 0 \quad v(4) = 0 \quad v(5) = 0 \tag{15}$$

3.1.2 Flows in the case of the coalition structure $\{1, 2\}, \{3, 4\}, 5$

If the players 3 and 4 merge into a coalition, the resulting flow rates will be as depicted in Fig. 3. This case represents a negative externality regarding the coalition $\{1, 2\}$.

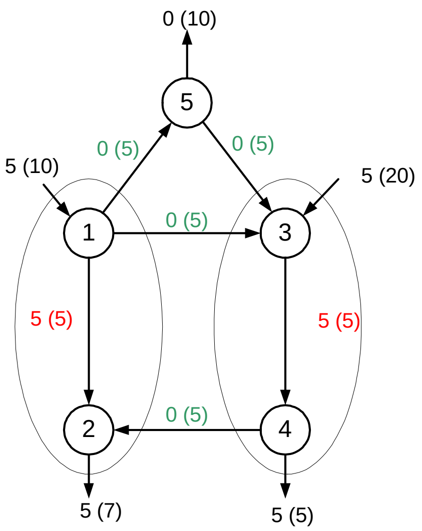

Figure 3: Flows in the case of the coalition structure $\{1, 2\}, \{3, 4\}, 5$.

As we can see, in this case the network is "horizontally" balanced, there is no power flow except on the lines connecting $1, 2$ and $3, 4$.

The μ, and v values in this case will be as follows

$$\mu(1) = 5 \quad \mu(2) = 5 \quad \mu(3) = 5 \quad \mu(4) = 5 \quad \mu(5) = 0 \tag{16}$$
$$v(\{1, 2\}) = 5 + 5 = 10 \quad v(\{3, 4\}) = 5 + 5 = 10 \quad v(5) = 0 \tag{17}$$

3.1.3 Flows in the case of the coalition structure $\{1, 2\}, \{3, 4, 5\}$

If the coalition $\{3, 4\}$ merges with player 5, the resulting configuration will give rise to transmission conditions depicted in Fig. 4. This serves as a reference case of a positive externality regarding the coalition $\{1, 2\}$.

In this case, also the horizontal edges of the network are utilized for power transmission, which enables a higher resulting transmission rate also for the coalition $\{1, 2\}$.

The μ, and v values in this case will be as follows

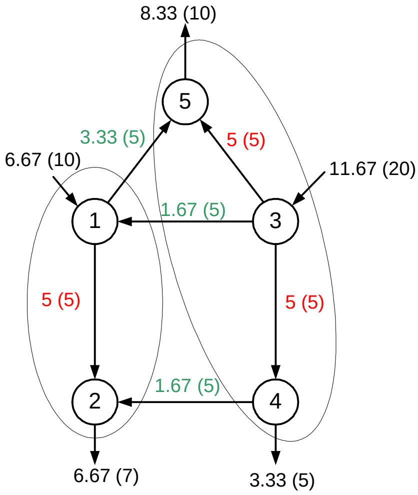

Figure 4: Flows in the case of the coalition structure $\{1, 2\}, \{3, 4, 5\}$.

$$\mu(1) = 6.67 \quad \mu(2) = 6.67 \quad \mu(3) = 11.67 \quad \mu(4) = 3.33 \quad \mu(5) = 8.33 \tag{18}$$

$$v(\{1, 2\}) = 6.67 + 6.67 = 13.33 \quad v(\{3, 4, 5\}) = 11.67 + 3.33 + 8.33 = 23.33 \tag{19}$$

3.2 Summary of the game

In table 1, the characteristic functions for the various partitions of the previously defined game (on the network depicted in Fig. 1) are given.

Table 1: Partition function of the PFF game

$\mathcal{P}$	V	$\mathcal{P}$	V
1,2,3,4,5	40	1,4 + 2,3,5	10 , 23.3333
1,2,3,4 + 5	20 , 0	1,4 + 2,3 + 5	10 , 10 , 0
1,2,3,5 + 4	34 , 0	1,5 + 2,3,4	20 , 20
1,2,3 + 4,5	14 , 0	1 + 2,3,4,5	0 , 28.8
1,2,3 + 4 + 5	14 , 0 , 0	1 + 2,3,4 + 5	0 , 19 , 0
1,2,4,5 + 3	20 , 0	1,5 + 2,3 + 4	19.7143 , 14 , 0
1,2,4 + 3,5	20 , 20	1 + 2,3,5 + 4	0 , 25.7143 , 0
1,2,4 + 3 + 5	17.5 , 0 , 0	1 + 2,3 + 4,5	0 , 14 , 0
1,2,5 + 3,4	20 , 10	1 + 2,3 + 4 + 5	0 , 14 , 0 , 0
1,2 + 3,4,5	13.3333 , 23.3333	1,4,5 + 2 + 3	20 , 0 , 0
1,2 + 3,4 + 5	10 , 10 , 0	1,4 + 2,5 + 3	10 , 0 , 0
1,2,5 + 3 + 4	20 , 0 , 0	1,4 + 2 + 3,5	10 , 0 , 18.5714
1,2 + 3,5 + 4	14 , 17.7143 , 0	1,4 + 2 + 3 + 5	10 , 0 , 0 , 0
1,2 + 3 + 4,5	13.75 , 0 , 0	1,5 + 2,4 + 3	15.7143 , 0 , 0
1,2 + 3 + 4 + 5	13.75 , 0 , 0 , 0	1 + 2,4,5 + 3	0 , 0 , 0
1,3,4,5 + 2	30 , 0	1 + 2,4 + 3,5	0 , 0 , 15.7143
1,3,4 + 2,5	10 , 0	1 + 2,4 + 3 + 5	0 , 0 , 0 , 0
1,3,4 + 2 + 5	10 , 0 , 0	1,5 + 2 + 3,4	17.1429 , 0 , 10
1,3,5 + 2,4	20 , 0	1 + 2,5 + 3,4	0 , 0 , 10
1,3 + 2,4,5	0 , 0	1 + 2 + 3,4,5	0 , 0 , 24.2857
1,3 + 2,4 + 5	0 , 0 , 0	1 + 2 + 3,4 + 5	0 , 0 , 10 , 0
1,3,5 + 2 + 4	20 , 0 , 0	1,5 + 2 + 3 + 4	15.7143 , 0 , 0 , 0
1,3 + 2,5 + 4	0 , 0 , 0	1 + 2,5 + 3 + 4	0 , 0 , 0 , 0
1,3 + 2 + 4,5	0 , 0 , 0	1 + 2 + 3,5 + 4	0 , 0 , 15.7143 , 0
1,3 + 2 + 4 + 5	0 , 0 , 0 , 0	1 + 2 + 3 + 4,5	0 , 0 , 0 , 0
1,4,5 + 2,3	20 , 14	1 + 2 + 3 + 4 + 5	0 , 0 , 0 , 0 , 0

3.3 Stability

The next step is to calculate the recursive core. Firstly observe that the game in this example is superadditive: the merger of two coalitions – assuming that other coalitions do not change – leads to an increase of the total payoffs (see table 1). This property is not true in general, but here it facilitates the calculation of the recursive core. For instance it is sufficient to study single-coalition deviations as multi-coalition deviations can never do better. As a further restriction only coalitions with at least one generator and one consumer are interesting. This rules out, for instance, singletons as potential deviators. The value of four player coalitions is directly given, since the remaining, fifth player can only form a singleton. For 3-player coalitions the situation is still simple, but less trivial: the remaining two players can remain singletons or form a pair. Due to the observed superadditivity they form a pair in all cases when the residual game is nontrivial, that is, when it consists of a generator and a consumer. Unfortunately in the "trivial" cases both partitions are feasible, and due to pessimism the one offering a lower payoff for the deviating triple is chosen. The deviating pairs constitute the most interesting case as the remaining 3 players can form 5 different partitions. Due to the observed superadditivty, however, a nonempty residual core will use the grand coalition as partition for all nontrivial residual games. Here the nonemptiness of the residual cores is nontrivial, but can easily be verified using balancedness Bondareva (1963); Shapley (1967).

Table 2: Characteristic function based on the PFF game

S	$v(S)$	S	$v(S)$
$\varnothing$	0	$\{1,2,3,4,5\}$	40
$\{1\}$	0	$\{2,3,4,5\}$	20
$\{2\}$	0	$\{1,3,4,5\}$	30
$\{3\}$	0	$\{1,2,4,5\}$	20
$\{4\}$	0	$\{1,2,3,5\}$	34
$\{5\}$	0	$\{1,2,3,4\}$	20
$\{1,2\}$	13.33	$\{3,4,5\}$	23.33
$\{1,3\}$	0	$\{2,4,5\}$	0
$\{1,4\}$	10	$\{2,3,5\}$	23.33
$\{1,5\}$	20	$\{2,3,4\}$	20
$\{2,3\}$	14	$\{1,4,5\}$	20
$\{2,4\}$	0	$\{1,3,5\}$	20
$\{2,5\}$	0	$\{1,3,4\}$	10
$\{3,4\}$	10	$\{1,2,5\}$	20
$\{3,5\}$	20	$\{1,2,4\}$	20
$\{4,5\}$	0	$\{1,2,3\}$	14

Given the characteristic function in Table 2 we can calculate the core. The core consists of all imputations x such that for all $S \subseteq N$ we have $\sum_{i \in S} x_i \geq v(S)$. While in general calculating the core is a complex problem, in this particular example the calculation is relatively straightforward.

First notice that the total payoff must be at least 40, which is also the payoff under the most efficient partitions: $\{\{1,2,3,4,5\}\}$, $\{\{1,5\},\{2,3,4\}\}$, $\{\{1,2,4\},\{3,5\}\}$. Clearly the payoffs of these coalitions will be exactly their characteristic value. After the simplification of the system of inequalities we find that the core collects the following points: The projection onto the $x_3 = x_4 = x_5 = 0$ plane is of the form of a parallelogram with vertices (4,

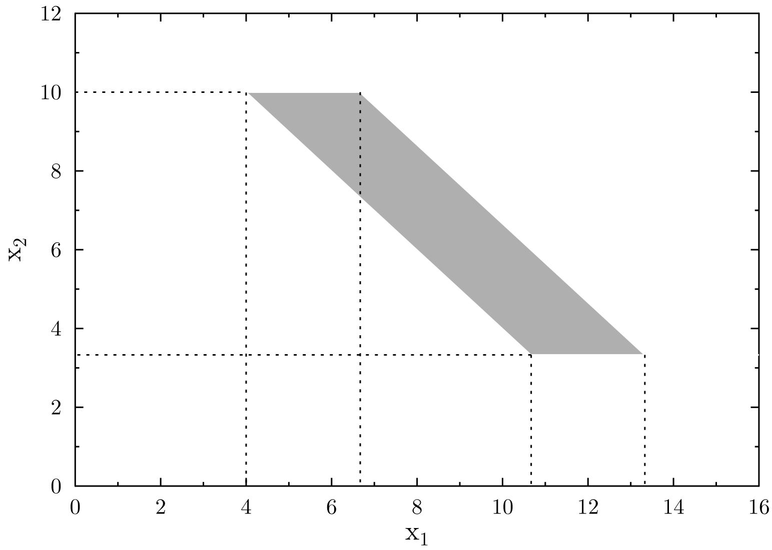

Figure 5: The projection of the core onto the $x_3 = x_4 = x_5 = 0$ plane.

10), (6.67, 10), (10.67, 3.33), (13.33, 3.33), while $x_3 = x_1$, $x_4 = 20 - x_1 - x_2$ and $x_5 = 40 - x_1 - x_2 - x_3 - x_4$.

When looking at the core notice that the equilibrium payoffs can vary greatly. In this game profitable coalition requires both generators and consumers and the value is produced jointly. This situation shows some resemblance to the glove games where players own left or right gloves and the value of a coalition is given by the number of pairs it owns. As one of the empirical criticisms of the core, it can be shown that the halves that are fewer may keep all the profit even if the difference in quantities is arbitrarily small when compared to the number of pairs. Here transmission capacity constraints and other physical properties of the network prevent the exploitation of one side or the other. This also means that the system of balancing groups remains most likely stable in the core sense even if legal regulations put restrictions on the distribution of profits.

3.4 Superadditivity

If $\mathcal{P}$ and $\mathcal{Q}$ are partitions, and $(\forall P \in \mathcal{P})(\exists Q \in \mathcal{Q})(P \subseteq Q)$, we say that $\mathcal{P}$ is a refinement of $\mathcal{Q}$. In this case, under superadditivity we mean that

$v(P_1,\mathcal{P})+...+v(P_k,\mathcal{P})\leq v(Q,\mathcal{Q})$.

Superadditivity is a natural property here, since the merger of two or more coalitions removes some of the constraints in the LP problem 12 thereby increasing the total flow. It turns out that these benefits do not necessarily stay within the merged coalitions, in fact they may be worse off after the merger!

In the following we will show on an example 6 node network that the defined game is not necessary superadditive. If two coalitions merge, it is possible that they generate a positive externality, meanwhile in turn their overall transmission will be reduced. The 6 node network used for the demonstration of this subadditive property is depicted in Fig. 6

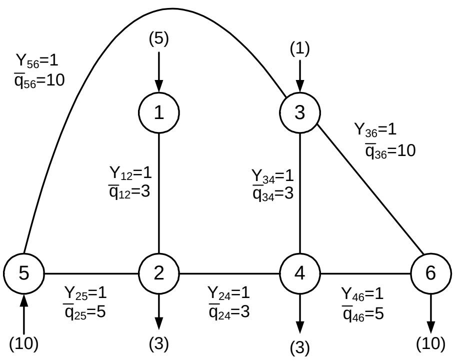

Figure 6: The basic structure, line and node parameters of the 6 node network

3.4.1 Flows in the case of the coalition structure $\{1,2\},\{3,4\},\{5,6\}$

The flows in the case of the coalition structure $\{1,2\},\{3,4\},\{5,6\}$ are depicted in Fig 7.

$$\begin{aligned}&\mu(1)=3\quad \mu(2)=3\quad \mu(3)=1\quad \mu(4)=1\quad \mu(5)=7.75\quad \mu(6)=7.75\\ &v(\{1,2\})=3+3=6\quad v(\{3,4\})=1+1=2\quad v(\{5,6\})=7.75+7.75=15.5\end{aligned}\tag{20}$$

Let us note that the coalitions $\{1,2\}$ and $\{3,4\}$ put load on the network only in vertical directions. The horizontal transfers can be related to the coalition $\{5,6\}$.

3.4.2 Flows in the case of the coalition structure $\{1,2,3,4\},\{5,6\}$

The resulting flows, corresponding to the scenario if the coalitions $\{1,2\}$ and $\{3,4\}$ merge, are depicted in Fig 8.

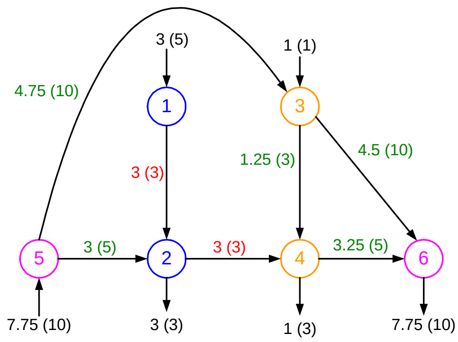

Figure 7: Power in and outlets and line flows in the case of the coalition structure $\{1, 2\}, \{3, 4\}, \{5, 6\}$. Coalitions are labeled with different colors.

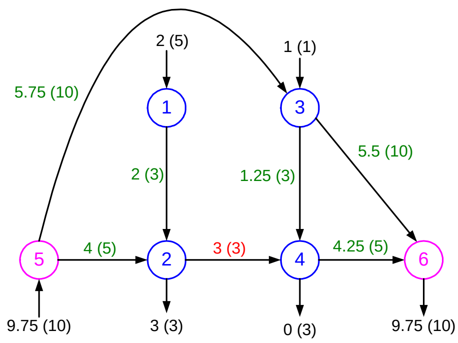

Figure 8: Power in and outlets and line flows in the case of the coalition structure $\{1, 2, 3, 4\}, \{5, 6\}$. Coalitions are labeled with different colors.

$$\mu(1) = 2 \quad \mu(2) = 3 \quad \mu(3) = 1 \quad \mu(4) = 0 \quad \mu(5) = 9.75 \quad \mu(6) = 9.75 \tag{21}$$
$$v(\{1, 2, 3, 4\}) = 2 + 3 + 1 = 6 \quad v(\{5, 6\}) = 9.75 + 9.75 = 19.5$$

As we see, as the coalitions $\{1, 2\}$ and $\{3, 4\}$ have merged, their overall value decreased from 6+2=8 to 6. Meanwhile, this reconfiguration of coalitions implied a positive externality of value 4 regarding the coalition $\{5, 6\}$. In this case of subadditivity, the positive externality is necessary, because in the case of less coalitions (which implies less constraints in the LP problem), the resulting overall network flow cannot be lower than in the case of more coalitions.

3.4.3 Discussion

What is the phenomenon behind this subadditivity example? If two coalitions merge, the number of the corresponding constraints is reduced. In the new configuration, they might put load on certain lines, on which they could not before. In this case this critical line is the line between node 2 and 4.

Until the coalition reconfiguration, the third generator-consumer pair ($\{5,6\}$) has used this line to transfer energy from left to right. This coalition also uses the route 5-3-6 with parallel to the route 5-2-4-6 for the transfer, however, the bottleneck is line 2-4.

As the constraints resolve, coalition $\{1,2,3,4\}$ can be 'forced' by the independent network regulator to use the line 2-4 for transfer from right to left (which implies the relief of line 2-4 from the point of view of $\{5,6\}$, who use it in the opposite direction). Furthermore, because the total transmitted energy of coalition $\{5,6\}$ is routed only partially via the line 2-4, they can increase their total transmitted amount more, compared to the quantity coalition $\{1,2,3,4\}$ had to reduced theirs to appropriately balance the line 2-4.

This way the total amount of power transmitted by the network increases, while the total amount of power transmission between nodes $\{1,2,3,4\}$ is reduced.

3.5 Changing the power grid

A straightforward way, in which we can extend the previously defined game of generator rescheduling, is the expansion problem. In this case, we will assume that there is a given set of possible line additions in the network, as depicted in the case of an example network in Fig. 9.

In this case, we assume that every line addition has a fixed cost (in this case 5 units). If a new line is added to the network, the resulting admittance and capacity of the line between the corresponding nodes will be the sum of the values of the new and the original lines (both admittance and capacity values are summarized).

Furthermore, we assume that a new line can be built only between two such nodes, which are in the same coalition (in the coalition which covers the cost of the line expansion). In this case, the cost of the line addition is naturally covered by the corresponding coalition. A line is built only, if it brings additional value for the coalition. Regarding the characteristic functions, for a given coalition C in the case of a certain partitioning, the value of v can be determined as follows: $v(C) = \max\{v_i(C)\}$ where the values $v_i(C)$ correspond to the possible line additions of the coalition (including also the case of no line addition). If additional lines are added to the network, $v_i(C)$ can

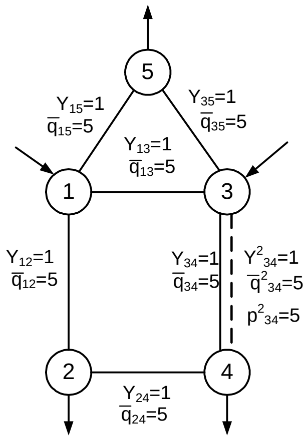

Figure 9: The basic structure and line parameters of the 5 node network in the case of a possible line extension between nodes 3 and 4 (denoted by the dashed line). p_{34}^2 denotes the cost of the line expansion.

be calculated as $v_i(C) = \sum_{i \in C} \mu(i) - \sum_j p_j$, where $\sum_j p_j$ describes the overall cost of actual line addition configuration. The values μ_i are determined corresponding to the network capacities and admittances corresponding to the network including the newly added lines.

3.5.1 Example for the expansion game

In this simple example, we analyze in the case of the simple 5 node network depicted in Fig. 9 which partitions imply the addition of a new line to the network. Let us assume that the maximal generator capacities and ideal consumption values are the same as described previously in section 3.1, except that the ideal consumption value of node 4 is 7.

Coalition structure $\{1, 2\}, \{3, 4\}, 5$
First, let us consider the coalition structure $\{1, 2\}$, $\{3, 4\}$, 5. We calculate all v_i values for the coalition $\{3, 4\}$, which is capable of a line addition to determine, whether the new line will be added in this coalition structure. If we consider no line addition (and denote this possibility with v_1), the flows will be the same as earlier (depicted in Fig. 3), so the value of v_1 can be calculated as $v_1(\{3, 4\}) = 5 + 5 = 10$ (the ideal consumption value of node 3 is higher compared to Fig. 3, but the transmission capacity constrains, which limit the transferable quantity are the same). If we assume the addition of a new line, the resulting power inlets/consumtions and flows will be as depicted

in Fig. 10

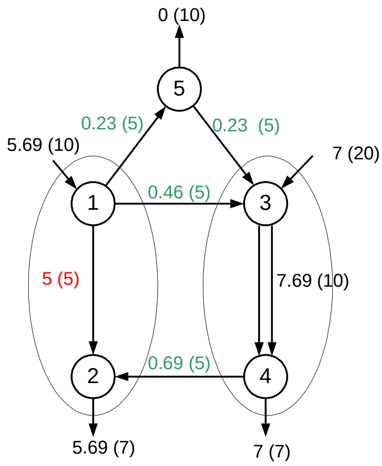

Figure 10: The power in/outlets and line flows of the network, in the case of the coalition structure $\{1,2\}$, $\{3,4\}$, 5, assuming that the line between node 3 and 4 is built

In this case

$$\begin{aligned} &\mu(1) = 5.69 \quad \mu(2) = 5.69 \quad \mu(3) = 7 \quad \mu(4) = 7 \quad \mu(5) = 0 \\ &v(\{1,2\}) = 5.69 + 5.69 = 11.38 \\ &v_2(\{3,4\}) = \sum_{i \in C} \mu(i) - p_{34}^2 = 7 + 7 - 5 = 9 \quad v(5) = 0 \end{aligned} \tag{22}$$

As we see $v_2(\{3,4\}) < v_1(\{3,4\})$. This means that the cost of the new line addition ($p_{34}^2 = 5$) exceeds the benefit that is implied by the enhanced power generation of node 3, and the higher consumption of node 4 (overall 2+2=4). This means, that in the case of this coalition structure, the new line between node 3 and 4 will not be built.

Coalition structure $\{1,2\},\{3,4,5\}$
Similarly to the previous case, if we consider no line addition ($v_1(\{3,4,5\})$), the flows will be the same as earlier (depicted in Fig. 4) in the case of this coalition structure, so the value of v_1 can be calculated as $v_1(\{3,4,5\}) = 11.67 + 3.33 + 8.33 = 23.33$. If we assume the addition of a new line, the resulting power inlets/consumptions and flows will be as depicted in Fig. 11.

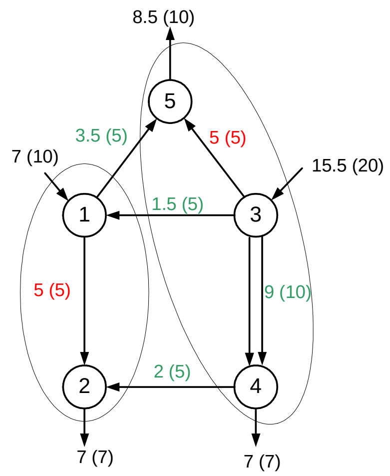

Figure 11: The power in/outlets and line flows of the network, in the case of the coalition structure $\{1,2\}$, $\{3,4,5\}$, assuming that the line between node 3 and 4 is built

In this case

$$
\begin{aligned}
&\mu(1) = 7 \quad \mu(2) = 7 \quad \mu(3) = 15.5 \quad \mu(4) = 7 \quad \mu(5) = 8.5 \\
&v(\{1,2\}) = 7 + 7 = 14 \\
&v_2(\{3,4,5\}) = \sum_{i \in C} \mu(i) - p_{34}^2 = 15.5 + 7 + 8.5 - 5 = 26
\end{aligned} \tag{23}
$$

In this case, as we see from $v_2(\{3,4,5\}) > v_1(\{3,4,5\})$,the overall effect of the line addition is beneficial for the coalition $\{3,4,5\}$, so the new line will be built in this case.

4 Conclusion

4.1 Summary

In this paper a cooperative game of generator rescheduling on electrical energy transmission networks has been proposed. The game is defined in a partition function form; for each partition, the characteristic function is given by the actual resulting power in and outflows that are calculated by an independent network regulator. This regulator optimizes the total transmission of the system, according to the line transmission capacity constraints and balance equations implied by the actual coalition structure. The optimization problem of the independent network regulator is formalized and solved as a linear programming problem.

We have shown that the game can give rise both to positive and negative externalities, and demonstrated this using a 5 node example network. In the case of the proposed superadditive example demonstrating externalities, the pessimistic recursive core has been calculated, and its non-emptiness for this example has been shown.

A second example of a 6 node network demonstrated that power transmission games are not necessary superadditive.

Furthermore, we described a straightforward method, due which the network expansion problem can be included in the proposed formalism.

4.2 Future work

Our model has been designed to be realistic enough to be interesting, but remain manageable. Some of the aspects of the real power transmission networks has been ignored for simplicity. Such are the redundancies to ensure that the network remains stable even in the case of an instant failure of any one transmission line (if we 'cut' one line, the resulting power flows must not exceed the transmission capacities anywhere in the network). This property can be included in the LP optimization problem, but it is not trivial how it will affect the properties of the resulting game. Reintroducing these elements will naturally add to the complexity to the model, but are unlikely to drastically alter our conclusions.

Regarding the dynamical extension of the game, two straightforward scenarios can be analyzed. One is the extension of the model in the direction of the description of daily change in energy demands, while the other corresponds to the network extension problem. In the case of daily adaptation, the analysis of safe (and efficient) transition between various network configuration can be a question of high interest. In the case of the extension problem, we may analyze for example whether a coalition, which builds a new line remains stable during a reorganization of coalitions according to the new network structure; whether it is beneficial to the linked parties and even if yes: who should stand the bill?

The steady increase of demand for energy and the increasing connectivity of national power networks are just two long-term trends that require large scale expansions. Ambitious plans will often boil down to the same question. A complex game theoretic model as ours that studies both the competition and cooperation of the parties can help to evaluate such conflicts.

References

Aumann, R. J., and B. Peleg (1960): “Von Neumann-Morgenstern Solutions to Cooperative Games Without Side Payments,” *Bulletin of the American Mathematical Society*, 66, 173–179.

Bai, X., S. Shahidehpour, and V. Ramesh (1997): “Transmission Analysis by Nash Game Method,” *IEEE Transactions on Power Systems*, 12, 1046–1052.

Bondareva, O. N. (1963): “Some Applications of Linear Programming Methods to the Theory of Cooperative Games,” *Problemy Kybernetiki*, 10, 119–139, (In Russian.).

Cardell, J. B., C. C. Hitt, and W. W. Hogan (1997): “Market power and strategic interaction in electricity networks,” *Resource and Energy Economics*, 19(1-2), 109–137.

Chander, P., and H. Tulkens (1997): “The core of and economy with multilateral environmental externalities,” *International Journal of Game Theory*, 26(3), 379–401.

Chen, Y., B. Hobbs, S. Leyffer, and T. S. Munson (2006): “Leader-follower equilibria for electric power and NO_x allowances markets,” *Computational Management Science*, 3, 307–330.

Contreras, J. (1997): “A Cooperative Game Theory Approach to Transmission Planning in Power Systems,” Ph.D. thesis, University of California, Berkeley.

Érdi, P. (2008): *Complexity Explained.* Springer, Berlin.

Evans, F., J. Zolezzi, and H. Rudnick (2003): “Cost assignment model for electrical transmission system expansion: an approach through the Kernel theory,” *IEEE Transactions on Power Systems*, 18, 625–632.

Gately, D. (1974): “Sharing the Gains from Regional Cooperation: A Game Theoretic Application to Planning Investment in Electric Power,” *International Economic Review*, 15, 195–208.

Gilbert, R., K. Neuhoff, and D. Newbery (2004): “Allocating Transmission to Mitigate Market Power in Electricity Networks,” *RAND Journal of Economics*, 35(4), 691—-709.

Gillies, D. B. (1959): "Solutions to general non-zero-sum games," in *Contributions to the Theory of Games IV*, ed. by A. W. Tucker, and R. D. Luce, no. 40 in Annals of Mathematics Studies, pp. 47–85. Princeton University Press, Princeton.

Hobbs, B. (1992): "Using game theory to analyze electric transmission pricing policies in the United States," *European Journal of Operational Research*, 56, 154–171.

Kirschen, D., and G. Strbac (2004): *Fundamentals of Power System Economics*. John Wiley & Sons, Ltd, Chichester, UK.

Kleindorfer, P., D.-J. Wu, and C. Fernando (2001): "Strategic gaming in electric power markets," *European Journal of Operational Research*, 130, 156–168.

Kóczy, L. Á. (2007): "A recursive core for partition function form games," *Theory and Decision*, 63(1), 41–51.

——— (2009): "Sequential Coalition Formation and the Core in the Presence of Externalities," *Games and Economic Behavior*, 66(1), 559–565.

Neuhoff, K., J. Barquin, M. Boots, A. Ehrenmann, B. Hobbs, F. Rijkers, and M. Vázquez (2005): "Network-constrained Cournot models of liberalized electricity markets: the devil is in the details," *Energy Economics*, 27, 495–525.

Oren, S., P. Spiller, P. Varaiya, and F. Wu (1995): "Folk Theorems on Transmission Access: Proofs and Counter Examples," Working papers series of the Program on Workable Energy Regulation (POWER) PWP-023, University of California Energy Institute 2539 Channing Way Berkeley, California 94720-5180, www.ucei.berkeley.edu/ucei.

Orths, A., A. Schmidtt, Z. Styczynski, and J. Verstege (2001): "Multi-Criteria optimization methods for planning and operation of electrical energy systems," *Electrical Engineering*, 83, 251–258.

Shapley, L. S. (1967): "On Balanced Sets and Cores," *Naval Research Logistics Quarterly*, 14(4), 453–460.

Thrall, R., and W. Lucas (1963): "n-person games in partition function form," *Naval Research Logistics Quarterly*, 10, 281–298.